Parallel addition
in non-standard numeration systems

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Algorithm of Avizienis 1961

Base $\beta = b$, $b \geq 3$ integer, parallel addition on alphabet
$A = \{-a, \ldots, 0, \ldots, a\}$, $b/2 < a \leq b - 1$.

**Input:** $x_n \cdots x_m$ and $y_n \cdots y_m$ in $A^*$, $m \leq n$,
$x = \sum_{i=m}^{n} x_i \beta^i$ and $y = \sum_{i=m}^{n} y_i \beta^i$.
**Output:** $z_{n+1} \cdots z_m$ in $A^*$ such that
$$z = x + y = \sum_{i=m}^{n+1} z_i \beta^i.$$  

for each $i$ in parallel do
0. $z_i := x_i + y_i$
1. if $z_i \geq a$ then $q_i := 1$, $r_i := z_i - b$
   if $z_i \leq -a$ then $q_i := -1$, $r_i := z_i + b$
   if $-a + 1 \leq z_i \leq a - 1$ then $q_i := 0$, $r_i := z_i$
2. $z_i := q_{i-1} + r_i$
Algorithm of Chow and Robertson 1978

Base $\beta = b = 2a$, $a \geq 1$, parallel addition on $A = \{-a, \ldots, 0, \ldots, a\}$.

**Input**: $x_n \cdots x_m$ and $y_n \cdots y_m$ in $A^*$, $m \leq n$, $x = \sum_{i=m}^{n} x_i \beta^i$ and $y = \sum_{i=m}^{n} y_i \beta^i$.

**Output**: $z_{n+1} \cdots z_m$ in $A^*$ such that $z = x + y = \sum_{i=m}^{n+1} z_i \beta^i$.

for each $i$ in parallel do
0. $z_i := x_i + y_i$
1. if $a + 1 \leq z_i \leq b$ then $q_i := 1$, $r_i := z_i - b$
   if $-b \leq z_i \leq -a - 1$ then $q_i := -1$, $r_i := z_i + b$
   if $-a + 1 \leq z_i \leq a - 1$ then $q_i := 0$, $r_i := z_i$
   if $z_i = a$ and $z_{i-1} > 0$ then $q_i := 1$, $r_i := -a$
   if $z_i = a$ and $z_{i-1} \leq 0$ then $q_i := 0$, $r_i := a$
   if $z_i = -a$ and $z_{i-1} < 0$ then $q_i := -1$, $r_i := a$
   if $z_i = -a$ and $z_{i-1} \geq 0$ then $q_i := 0$, $r_i := -a$
2. $z_i := q_{i-1} + r_i$
Excursion into symbolic dynamics

$\mathcal{A}^\mathbb{Z}$ equipped with the shift is a symbolic dynamical system (the full shift).

In computer arithmetic (Kornerup):

$\varphi : \mathcal{A}^\mathbb{Z} \to \mathcal{B}^\mathbb{Z}$ is \textit{computable in parallel} if $\exists r, t > 0$, $\exists k \geq 0$, and $\exists \Phi : \mathcal{A}^p \to \mathcal{B}^k$, with $p = r + t + k$, such that if $u = (u_i)_{i \in \mathbb{Z}} \in \mathcal{A}^\mathbb{Z}$ and $v = (v_i)_{i \in \mathbb{Z}} \in \mathcal{B}^\mathbb{Z}$, then

$$v = \varphi(u) \iff \forall i \in \mathbb{Z}, \ v_{ki+k-1} \cdots v_{ki} = \Phi(u_{ki+k+t-1} \cdots u_{ki-r}).$$

The image of $u$ by $\varphi$ is obtained through a sliding window of length $p$. 
In symbolic dynamics: $S \subseteq A^\mathbb{Z}$ and $T \subseteq B^\mathbb{Z}$ symbolic dynamical systems. \( \varphi : S \rightarrow T \) is a local function if it is a function computable in parallel with \( k = 1 \). It is called a sliding block code. \( r \) is the memory and \( t \) is the anticipation of \( \varphi \). The function \( \varphi \) is said to be \( p \)-local.

Remark

A \( p \)-local function from \( A^\mathbb{Z} \) to \( B^\mathbb{Z} \) is realizable by a finite on-line transducer with delay \( p - 1 \).
In symbolic dynamics: 
\( S \subseteq A^\mathbb{Z} \) and \( T \subseteq B^\mathbb{Z} \) symbolic dynamical systems. 
\( \varphi : S \rightarrow T \) is a \textit{local function} if it is a function computable in parallel with \( k = 1 \). It is called a \textit{sliding block code}. 
\( r \) is the \textit{memory} and \( t \) is the \textit{anticipation} of \( \varphi \). 
The function \( \varphi \) is said to be \textit{p-local}. 

\textbf{Remark} 

A \textit{p-local function} from \( A^\mathbb{Z} \) to \( B^\mathbb{Z} \) is realizable by a \textit{finite on-line transducer with delay} \( p – 1 \). 

Addition is a function from \( \{−2a, \ldots, 0, \ldots, 2a\}^\mathbb{Z} \) to \( \{−a, \ldots, 0, \ldots, a\}^\mathbb{Z} \).
Differences between the two algorithms

Decision (choice) in step 1:

- Avizienis algorithm is neighbour free.
- Chow and Robertson algorithm is neighbour sensitive.

Locality

- Avizienis addition is 2-local.
- Chow and Robertson addition is 3-local.
Strong representation of zero property

Base $\beta$ algebraic number with $|\beta| > 1$.

**Definition**

$\beta$ satisfies the *strong representation of zero property* ($\beta$ is SRZ) if there exist integers $b_k, b_{k-1}, \ldots, b_1, b_0, b_{-1}, \ldots, b_{-h}$, with $h$ and $k$ non-negative integers, such that $\beta$ is a root of the polynomial

$$S(X) = b_k X^k + b_{k-1} X^{k-1} + \cdots + b_1 X + b_0 + b_{-1} X^{-1} + \cdots + b_{-h} X^{-h}$$

and

$$B = b_0 > 2 \sum_{i \neq 0} |b_i| = 2M.$$  

The polynomial $S$ is said to be a *strong polynomial* for $\beta$. 
Strong representation of zero property

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and

$$B = b_0 > 2 \sum_{i \neq 0} |b_i| = 2M.$$

The polynomial $S$ is said to be a *strong polynomial* for $\beta$.

$$(b_k b_{k-1} \cdots b_1 b_0 b_{-1} \cdots b_{-h})_{\beta} = 0$$
Suppose that $\beta$ is SRZ, i.e. $B > 2M$.
Working alphabet $\mathcal{A} = \{-a, \ldots, 0, \ldots, a\}$ with
$$a = \left\lfloor \frac{B-1}{2} \right\rfloor + \left\lfloor \frac{B-1}{2(B-2M)} \right\rfloor M.$$
Let
$$a' = \left\lfloor \frac{B-1}{2} \right\rfloor \quad \text{and} \quad c = \left\lfloor \frac{B-1}{2(B-2M)} \right\rfloor.$$
Then $a = a' + cM$.
$\mathcal{A}' = \{-a', \ldots, 0, \ldots, a'\} \subset \mathcal{A}$ is the inner alphabet.
Parallel addition for base $\beta$ SRZ on $A = \{-a, \ldots, 0, \ldots, a\}$, $a = \left\lfloor \frac{B-1}{2} \right\rfloor + \left\lfloor \frac{B-1}{2(B-2M)} \right\rfloor M$

Algorithm (S)

**Input:** $x_n \cdots x_m$ and $y_n \cdots y_m$ in $A^*$, with $m \leq n$, $x = \sum_{i=m}^{n} x_i \beta^i$ and $y = \sum_{i=m}^{n} y_i \beta^i$.

**Output:** $z_{n+k} \cdots z_{m-h}$ in $A^*$ such that $z = x + y = \sum_{i=m-h}^{n+k} z_i \beta^i$.

for each $i$ in parallel do

0. $z_i := x_i + y_i$

1. find $q_i \in \{-c, \ldots, 0, \ldots, c\}$ such that $z_i - q_i B \in A'$

2. $z_i := z_i - \sum_{j=-h}^{k} q_{i-j} b_j$
Parallel addition for base $\beta$ SRZ on $\mathcal{A} = \{-a, \ldots, 0, \ldots, a\}, \ a = \left[ \frac{B-1}{2} \right] + \left[ \frac{B-1}{2(B-2M)} \right] M$

Algorithm (S)

**Input:** $x_n \cdots x_m$ and $y_n \cdots y_m$ in $\mathcal{A}^*$, with $m \leq n$, $x = \sum_{i=m}^{n} x_i \beta^i$ and $y = \sum_{i=m}^{n} y_i \beta^i$.

**Output:** $z_{n+k} \cdots z_{m-h}$ in $\mathcal{A}^*$ such that $z = x + y = \sum_{i=m-h}^{n+k} z_i \beta^i$.

for each $i$ in parallel do

0. $z_i := x_i + y_i$
1. find $q_i \in \{-c, \ldots, 0, \ldots, c\}$ such that $z_i - q_i B \in \mathcal{A}'$

2. $z_i := z_i - \sum_{j=-h}^{k} q_{i-j} b_j$

Algorithm (S) is neighbour free.
$\beta = b \text{ integer } \geq 3$ is SRZ for the polynomial $-X + b$, and Algorithm (S) works with $c = 1$, $a' = \left\lceil \frac{b-1}{2} \right\rceil$, and $a = \left\lceil \frac{b+1}{2} \right\rceil$.

for each $i$ in parallel do
0. $z_i := x_i + y_i$
1. find $q_i \in \{-1, 0, 1\}$ such that $z_i - q_i b \in A'$
2. $z_i := z_i - q_i b + q_{i-1}$

Algorithm (S) is the algorithm of Avizienis with $a = \left\lceil \frac{b+1}{2} \right\rceil$. 
For $\beta = 2$, $-X + 2$ is not a strong polynomial. But $\beta$ satisfies the strong polynomial $-X^2 + 4$.

So Algorithm (S) works for base 2 on $\{-3, \ldots, 0, \ldots, 3\}$. 
For $\beta = 2$, $-X + 2$ is not a strong polynomial. But $\beta$ satisfies the strong polynomial 

$$-X^2 + 4.$$

So Algorithm (S) works for base 2 on $\{-3, \ldots, 0, \ldots, 3\}$.

Remind that the Chow and Robertson algorithm works with smaller alphabet $\{-1, 0, 1\}$, but need to examine the right neighbour of current position.
The Golden Mean

\[ \beta = \frac{1 + \sqrt{5}}{2}, \] the Golden Mean.
Every real number \( \geq 0 \) has an expansion on alphabet \( \{0, 1\} \).

\[ \beta \] is one root of \( X^2 - X - 1 \), the second root is \( \beta' = \frac{1 - \sqrt{5}}{2} = -\frac{1}{\beta}. \)
Since \( \beta^4 + (\beta')^4 = 7 \), \( \beta \) is a root of the strong polynomial

\[ S(X) = -X^4 + 7 - \frac{1}{X^4} \]

with \( B = 7 \) and \( M = 2 \). Thus \( c = 1, \ a' = 3 \), and \( a = 5 \). The working alphabet of Algorithm (S) is \( A = \{-5, \ldots, 0, \ldots, 5\} \).

1000700001 is a strong \( \beta \)-representation of 0.
\( a' = 3, \ a = 5 \)

| \( x \) | \( \mapsto \) | \( 2 \ 5 \ \bar{2} \ 5 \ \bar{5} \ 0 \ 0 \ 3 \) |
| \( y \) | \( \mapsto \) | \( 5 \ 1 \ 2 \ 2 \ 5 \ \bar{4} \ 0 \ 0 \ 5 \) |
| \( z \) | \( \mapsto \) | \( 5 \ 3 \ 7 \ \bar{4} \ 10 \ \bar{9} \ 0 \ 0 \ 8 \) |

| 0 | \( \mapsto \) | 1 0 0 0 7 0 0 0 1 |
| 0 | \( \mapsto \) | 1 0 0 0 7 0 0 0 1 |
| 0 | \( \mapsto \) | \( \bar{1} \ 0 \ 0 \ 0 \ 7 \ 0 \ 0 \ 0 \ 1 \) |
| 0 | \( \mapsto \) | 1 0 0 0 7 0 0 0 1 |
| 0 | \( \mapsto \) | \( \bar{1} \ 0 \ 0 \ 0 \ 7 \ 0 \ 0 \ 0 \ 1 \) |
| 0 | \( \mapsto \) | 1 0 0 0 7 0 0 0 1 |

\( z \mapsto 1 0 1 \ \bar{1} \ \bar{1} \ 2 \ 0 \ 3 \ 5 \ \bar{2} \ 1 \ \bar{1} \ 2 \ \bar{1} \ 0 \ 0 \ 1 \)
Example
Base $\beta = -1 + i$.
Every complex number has an expansion on alphabet $\{0, 1\}$.

$\beta$ satisfies the strong polynomial $X^4 + 4$ with $B = 4$ and $M = 1$. 
The working alphabet of Algorithm (S) is $A = \{-3, \ldots, 0, \ldots, 3\}$.
**Gaussian integer**

**Example**

Base $\beta = -1 + i$.

Every complex number has an expansion on alphabet \{0, 1\}.

$\beta$ satisfies the strong polynomial $X^4 + 4$ with $B = 4$ and $M = 1$.

The working alphabet of Algorithm (S) is $\mathcal{A} = \{-3, \ldots, 0, \ldots, 3\}$.

On \{-1, 0, 1\}, there is a parallel addition algorithm provided by Nielsen and Muller (1996).
Example

$\beta = \frac{7}{2}$ is an algebraic number which is not an algebraic integer. $\beta$ is SRZ for the polynomial $S(X) = -2X + 7$ with $B = 7$ and $M = 2$. Thus $a' = 3$, $c = 1$, and $a = 5$.

Any integer has a finite representation on the redundant alphabet $\{-5, \ldots, 0, \ldots, 5\}$ (Frougny and Klouda 2010), and, by Algorithm (S), addition is realizable in parallel.
Corollary

If $\beta$ is SRZ with strong polynomial $S(X) = b_kX^k + b_{k-1}X^{k-1} + \cdots + b_1X + b_0 + b_{-1}X^{-1} + \cdots + b_{-h}X^{-h}$ then addition realized by Algorithm (S) is a $(h + k + 1)$-local function from $\{-2a, \ldots, 0, \ldots, 2a\}^\mathbb{Z}$ to $A^\mathbb{Z}$. 
What numbers are SRZ?

**Proposition**

Let $\beta$ with $|\beta| > 1$ be an algebraic number such that all its algebraic conjugates have modulus $\neq 1$. Then $\beta$ is SRZ.

The proof gives a constructive method to obtain a strong polynomial from the minimal polynomial of $\beta$. 
What numbers are SRZ?

Proposition

Let $\beta$ with $|\beta| > 1$ be an algebraic number such that all its algebraic conjugates have modulus $\neq 1$. Then $\beta$ is SRZ.

The proof gives a constructive method to obtain a strong polynomial from the minimal polynomial of $\beta$.

Theorem

Let $\beta$ with $|\beta| > 1$ be an algebraic number of degree $d$.

$\square$ If $d = 2$, then $\beta$ is SRZ.

$\square$ If $d$ is even $\geq 4$ and the minimal polynomial of $\beta$ is not reciprocal, then $\beta$ is SRZ.

$\square$ If $d$ is odd, then $\beta$ is SRZ.
**Reduction of the alphabet**

**Definition**

$\beta$ satisfies the **weak representation of zero property** ($\beta$ is WRZ) if there exist integers $b_k, b_{k-1}, \ldots, b_1, b_0, b_{-1}, \ldots, b_{-h}$ with $h$ and $k$ non-negative integers, such that $\beta$ is a root of the polynomial

$$W(X) = b_k X^k + b_{k-1} X^{k-1} + \ldots + b_1 X + b_0 + b_{-1} X^{-1} + \ldots + b_{-h} X^{-h}$$

and

$$B = b_0 > \sum_{i \neq 0} |b_i| = M.$$  

The polynomial $W$ is said to be a **weak polynomial** for $\beta$. 
\( \beta \) is WRZ, i.e. \( B > M \). Working alphabet

\[
\mathcal{A} = \{-a, \ldots, 0, \ldots, a\}, \quad \text{where} \quad a = \left\lceil \frac{B-1}{2} \right\rceil + M.
\]

Inner alphabet is \( \mathcal{A}' = \{-a', \ldots, 0, \ldots, a'\} \) with \( a' = \left\lceil \frac{B-1}{2} \right\rceil \).

Algorithm (W) works with \( \left\lceil \frac{a}{B-M} \right\rceil \) iterations.
Parallel addition for base $\beta$ WRZ on $\mathcal{A} = \{-a, \ldots, 0, \ldots, a\}$, $a = \left\lfloor \frac{B-1}{2} \right\rfloor + M$

Algorithm (W)

Input: $x_n \cdots x_m$ and $y_n \cdots y_m$ in $\mathcal{A}^*$, with $m \leq n$, $x = \sum_{i=m}^{n} x_i \beta^i$ and $y = \sum_{i=m}^{n} y_i \beta^i$.

Output: $z_{n+k} \cdots z_{m-h}$ in $\mathcal{A}^*$ such that

$$z = x + y = \sum_{i=m-h}^{n+k} z_i \beta^i.$$

for each $i$ in parallel do

0. $z_i := x_i + y_i$

1. for $\ell := 1$ to $\left\lfloor \frac{a}{B-M} \right\rfloor$ do

   if $z_i \in \mathcal{A}'$ then $q_i := 0$ else $q_i := \text{sgn } z_i$

   $z_i := z_i - \sum_{j=-h}^{k} q_{i-j} b_j$
Example

$\beta = \frac{1 + \sqrt{5}}{2}$, the Golden Mean.
Since $-\beta^2 + 3 - \frac{1}{\beta^2} = 0$, $\beta$ is a root of the weak polynomial

$$W(X) = -X^2 + 3 - \frac{1}{X^2}$$

with $B = 3$ and $M = 2$. Thus $a' = 1$, and $a = 3$.
Algorithm (W) works on $A = \{-3, \ldots, 0, \ldots, 3\}$, with 3 iterations.

$\overline{10301}$ is a weak $\beta$-representation of 0.
<table>
<thead>
<tr>
<th>Step</th>
<th>x</th>
<th>y</th>
<th>z</th>
</tr>
</thead>
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<tr>
<td>Step 0.</td>
<td><img src="image1.png" alt="Image" /></td>
<td><img src="image2.png" alt="Image" /></td>
<td><img src="image3.png" alt="Image" /></td>
</tr>
<tr>
<td>Step 1.</td>
<td><img src="image4.png" alt="Image" /></td>
<td><img src="image5.png" alt="Image" /></td>
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<td><img src="image7.png" alt="Image" /></td>
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<td>Step 3.</td>
<td><img src="image10.png" alt="Image" /></td>
<td><img src="image11.png" alt="Image" /></td>
<td><img src="image12.png" alt="Image" /></td>
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</table>
Corollary

If \( \beta \) is WRZ with weak polynomial \( W(X) = b_k X^k + b_{k-1} X^{k-1} + \cdots + b_1 X + b_0 + b_{-1} X^{-1} + \cdots + b_{-h} X^{-h} \) then addition realized by Algorithm (W) is a \( (h \left\lceil \frac{a}{B-M} \right\rceil + k \left\lceil \frac{a}{B-M} \right\rceil + 1) \)-local function from \( \{ -2a, \ldots, 0, \ldots, 2a \} \mathbb{Z} \) to \( A^\mathbb{Z} \).
Corollary

If $\beta$ is WRZ with weak polynomial $W(X) = b_k X^k + b_{k-1} X^{k-1} + \cdots + b_1 X + b_0 + b_{-1} X^{-1} + \cdots + b_{-h} X^{-h}$ then addition realized by Algorithm (W) is a

$(h \left\lceil \frac{a}{B-M} \right\rceil + k \left\lceil \frac{a}{B-M} \right\rceil + 1)$-local function from $\{ -2a, \ldots, 0, \ldots, 2a \}^\mathbb{Z}$ to $A^\mathbb{Z}$.

Algorithm (W) is neighbour free.
Corollary

If \( \beta \) is WRZ with weak polynomial \( W(X) = b_kX^k + b_{k-1}X^{k-1} + \cdots + b_1X + b_0 + b_{-1}X^{-1} + \cdots + b_{-h}X^{-h} \) then addition realized by Algorithm \((W)\) is a \((h \left\lceil \frac{a}{B-M} \right\rceil + k \left\lceil \frac{a}{B-M} \right\rceil + 1)\)-local function from \(\mathbb{Z} \) to \(A^\mathbb{Z}\).

Algorithm (W) is neighbour free.

Remark

Algorithm (S) and Algorithm (W) coincide if, and only if, \(B \geq 4M - 1\).

Example

- If \(b\) integer \( \geq 3\), \(-X + b\) is a strong polynomial for \(b\), with \(B = b\) and \(M = 1\), thus \(B \geq 4M - 1\). Algorithm (S) and Algorithm (W) coincide with \(A = \{-a, \ldots, a\}, \ a = \left\lceil \frac{b+1}{2} \right\rceil\).

- For \(b = 2\), \(-X + 2\) is a weak polynomial. Algorithm (W) works with \(A = \{-2, -1, 0, 1, 2\}\) with 2 iterations.
The Golden Mean

In 1986 Berstel has given a parallel addition algorithm in base the Golden Mean on \{0, 1, \ldots, 12\}.
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In 1986 Berstel has given a parallel addition algorithm in base the Golden Mean on \{0, 1, \ldots, 12\}.

We give Algorithm (G) for parallel addition in base the Golden Mean on \{-1, 0, 1\}. This algorithm is neighbour sensitive.

It is not possible to perform parallel addition in base the Golden Mean on \{0, 1\}.
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In 1986 Berstel has given a parallel addition algorithm in base the Golden Mean on \{0, 1, \ldots, 12\}.

We give Algorithm (G) for parallel addition in base the Golden Mean on \{-1, 0, 1\}. This algorithm is neighbour sensitive.

It is not possible to perform parallel addition in base the Golden Mean on \{0, 1\}.

We use the weak representation of zero \(-\beta^2 + 3 - \frac{1}{\beta^2} = 0\).
Algorithm A: Base $\beta = \frac{1 + \sqrt{5}}{2}$, reduction from $\{-2, -1, 0, 1, 2\}$ to $\{-1, 0, 1, 2\}$.

**Input**: a finite sequence of digits $(z_i)$ of $\{-2, -1, 0, 1, 2\}$, with $z = \sum z_i \beta^i$.

**Output**: a finite sequence of digits $(z_i)$ of $\{-1, 0, 1, 2\}$, with $z = \sum z_i \beta^i$.

for each $i$ in parallel do

1. case
   
   \[
   \begin{cases}
   z_i = -2 \\
   z_i = -1 \\
   z_i = 0 \text{ and } z_{i+2} < 0 \text{ and } z_{i-2} < 0
   \end{cases}
   \]
   then $q_i := -1$

   else $q_i := 0$

2. $z_i := z_i - 3q_i + q_{i+2} + q_{i-2}$
Algorithm B: Base $\beta = \frac{1 + \sqrt{5}}{2}$, reduction from $\{-1, 0, 1, 2\}$ to $\{-1, 0, 1\}$.

Input: a finite sequence of digits $(z_i)$ of $\{-1, 0, 1, 2\}$, with $z = \sum z_i \beta^i$.

Output: a finite sequence of digits $(z_i)$ of $\{-1, 0, 1\}$, with $z = \sum z_i \beta^i$.

for each $i$ in parallel do
1. case
   \[ z_i = 2 \]
   \[ z_i = 1 \text{ and } (z_{i+2} \geq 1 \text{ or } z_{i-2} \geq 1) \]
   \[ z_i = 0 \text{ and } z_{i+2} = z_{i-2} = 2 \]
   \[ z_i = 0 \text{ and } z_{i+2} = z_{i-2} = 1 \text{ and } z_{i+4} \geq 1 \text{ and } z_{i-4} \geq 1 \]
   \[ z_i = 0 \text{ and } z_{i+2} = 2 \text{ and } z_{i-2} = 1 \text{ and } z_{i-4} \geq 1 \]
   \[ z_i = 0 \text{ and } z_{i-2} = 2 \text{ and } z_{i+2} = 1 \text{ and } z_{i+4} \geq 1 \]
   then $q_i := 1$

   else $q_i := 0$
2. $z_i := z_i - 3q_i + q_{i+2} + q_{i-2}$
Algorithm (G): Base \( \beta = \frac{1+\sqrt{5}}{2} \), parallel addition on \( A = \{-1, 0, 1\} \).

Input: two finite sequences of digits \((x_i)\) and \((y_i)\) of \(\{-1, 0, 1\}\), with \(x = \sum x_i \beta^i\) and \(y = \sum y_i \beta^i\).

Output: a finite sequence of digits \((z_i)\) of \(\{-1, 0, 1\}\) such that

\[
z = x + y = \sum z_i \beta^i.
\]

for each \(i\) in parallel do

0. \(v_i := x_i + y_i\)

1. use Algorithm A with input \((v_i)\) and output \((w_i)\)

2. use Algorithm B with input \((w_i)\) and output \((z_i)\)
Corollary

Addition in base the Golden Mean realized by Algorithm (G) is a 21-local function from $\{-2, -1, 0, 1, 2\}^\mathbb{Z}$ to $\{-1, 0, 1\}^\mathbb{Z}$. 
Some questions

A alphabet of contiguous integer digits.
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- What is the minimal symmetric alphabet $\mathcal{A} = \{-a, \ldots, 0, \ldots, a\}$ for parallel addition?
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- What is the minimal symmetric alphabet $\mathcal{A} = \{-a, \ldots, 0, \ldots, a\}$ for parallel addition?
  - If $\beta = b$ odd integer $\geq 3$, $a = \lceil \frac{b+1}{2} \rceil$ (Avizienis).
  - If $\beta = b$ even integer $\geq 2$, $a = \frac{b}{2}$ (Chow and Robertson).
  - If $\beta$ is the Golden Mean, $a = 1$. 
A alphabet of contiguous integer digits.

- What is the minimal symmetric alphabet $A = \{-a, \ldots, 0, \ldots, a\}$ for parallel addition?
  If $\beta = b$ odd integer $\geq 3$, $a = \left\lfloor \frac{b+1}{2} \right\rfloor$ (Avizienis).
  If $\beta = b$ even integer $\geq 2$, $a = \frac{b}{2}$ (Chow and Robertson).
  If $\beta$ is the Golden Mean, $a = 1$.

- What are the minimal alphabets for parallel addition?
Some questions

An alphabet of contiguous integer digits.

- **What is the minimal symmetric alphabet** $\mathcal{A} = \{-a, \ldots, 0, \ldots, a\}$ for parallel addition?
  - If $\beta = b$ odd integer $\geq 3$, $a = \lceil \frac{b+1}{2} \rceil$ (Avizienis).
  - If $\beta = b$ even integer $\geq 2$, $a = \frac{b}{2}$ (Chow and Robertson).
  - If $\beta$ is the Golden Mean, $a = 1$.

- **What are the minimal alphabets for parallel addition?**
  - If $\beta = b \geq 2$, an alphabet of cardinality $b + 1$: $\{-1, 0, \ldots, b - 1\}$, or $\{0, \ldots, b - 1, b\}$, ...