Parallel addition in non-standard numeration systems

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Algorithm of Avizienis 1961

Base $\beta = b$, $b \geqslant 3$ integer, parallel addition on alphabet $\mathcal{A} = \{-a, \ldots, 0, \ldots, a\}$, $b/2 < a \leqslant b - 1$.

Input: $x_n \cdots x_m$ and $y_n \cdots y_m$ in $\mathcal{A}^*$, $m \leqslant n$,
$x = \sum_{i=m}^{n} x_i \beta^i$ and $y = \sum_{i=m}^{n} y_i \beta^i$.
Output: $z_{n+1} \cdots z_m$ in $\mathcal{A}^*$ such that

$$z = x + y = \sum_{i=m}^{n+1} z_i \beta^i.$$

`for each` i `in parallel do`

0. $z_i := x_i + y_i$
1. `if` $z_i \geqslant a$ `then` $q_i := 1$, $r_i := z_i - b$
   `if` $z_i \leqslant -a$ `then` $q_i := -1$, $r_i := z_i + b$
   `if` $-a + 1 \leqslant z_i \leqslant a - 1$ `then` $q_i := 0$, $r_i := z_i$
2. $z_i := q_{i-1} + r_i$

Algorithm of Chow and Robertson 1978

Base $\beta = b = 2a$, $a \geqslant 1$, parallel addition on $\mathcal{A} = \{-a, \ldots, 0, \ldots, a\}$.

Input: $x_n \cdots x_m$ and $y_n \cdots y_m$ in $\mathcal{A}^*$, $m \leqslant n$,
$x = \sum_{i=m}^{n} x_i \beta^i$ and $y = \sum_{i=m}^{n} y_i \beta^i$.
Output: $z_{n+1} \cdots z_m$ in $\mathcal{A}^*$ such that $z = x + y = \sum_{i=m}^{n+1} z_i \beta^i$.

`for each` i `in parallel do`

0. $z_i := x_i + y_i$
1. `if` $a + 1 \leqslant z_i \leqslant b$ `then` $q_i := 1$, $r_i := z_i - b$
   `if` $-b \leqslant z_i \leqslant -a - 1$ `then` $q_i := -1$, $r_i := z_i + b$
   `if` $-a + 1 \leqslant z_i \leqslant a - 1$ `then` $q_i := 0$, $r_i := z_i$
   `if` $z_i = a$ `and` $z_{i-1} > 0$ `then` $q_i := 1$, $r_i := -a$
   `if` $z_i = a$ `and` $z_{i-1} \leqslant 0$ `then` $q_i := 0$, $r_i := a$
   `if` $z_i = -a$ `and` $z_{i-1} < 0$ `then` $q_i := -1$, $r_i := a$
   `if` $z_i = -a$ `and` $z_{i-1} \geqslant 0$ `then` $q_i := 0$, $r_i := -a$
2. $z_i := q_{i-1} + r_i$

Excursion into symbolic dynamics

$\mathcal{A}^{\mathbb{Z}}$ equipped with the shift is a symbolic dynamical system (the full shift).
In computer arithmetic (Kornerup):
$\varphi : \mathcal{A}^{\mathbb{Z}} \to \mathcal{B}^{\mathbb{Z}}$ is *computable in parallel* if $\exists r, t > 0$, $\exists k \geqslant 0$, and $\exists \Phi : \mathcal{A}^p \to \mathcal{B}^k$, with $p = r + t + k$, such that if $u = (u_i)_{i \in \mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$ and $v = (v_i)_{i \in \mathbb{Z}} \in \mathcal{B}^{\mathbb{Z}}$, then

$$v = \varphi(u) \iff \forall i \in \mathbb{Z}, \;\; v_{ki+k-1} \cdots v_{ki} = \Phi(u_{ki+k+t-1} \cdots u_{ki-r}).$$

The image of u by φ is obtained through a sliding window of length p.

In symbolic dynamics:
$S \subseteq \mathcal{A}^{\mathbb{Z}}$ and $T \subseteq \mathcal{B}^{\mathbb{Z}}$ symbolic dynamical systems.
$\varphi : S \to T$ is a *local function* if it is a function computable in parallel with $k = 1$. It is called a *sliding block code*.
r is the *memory* and t is the *anticipation* of φ.
The function φ is said to be *p-local*.

Remark

A p-local function from $\mathcal{A}^{\mathbb{Z}}$ to $\mathcal{B}^{\mathbb{Z}}$ is realizable by a finite on-line transducer with delay $p - 1$.

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A p-local function from $\mathcal{A}^{\mathbb{Z}}$ to $\mathcal{B}^{\mathbb{Z}}$ is realizable by a finite on-line transducer with delay $p - 1$.

Addition is a function from $\{-2a, \ldots, 0, \ldots, 2a\}^{\mathbb{Z}}$ to $\{-a, \ldots, 0, \ldots, a\}^{\mathbb{Z}}$.

Differences between the two algorithms

Decision (choice) in step 1:

- Avizienis algorithm is neighbour free.
- Chow and Robertson algorithm is neighbour sensitive.

Locality

- Avizienis addition is 2-local.
- Chow and Robertson addition is 3-local.

Strong representation of zero property

Base β algebraic number with $|\beta| > 1$.

Definition

β satisfies the *strong representation of zero property* (β *is SRZ*) if there exist integers $b_k, b_{k-1}, \ldots, b_1, b_0, b_{-1}, \ldots, b_{-h}$, with h and k non-negative integers, such that β is a root of the polynomial

$$S(X) = b_k X^k + b_{k-1} X^{k-1} + \cdots + b_1 X + b_0 + b_{-1} X^{-1} + \cdots + b_{-h} X^{-h}$$

and

$$B = b_0 > 2 \sum_{i \neq 0} |b_i| = 2M\,.$$

The polynomial S is said to be a *strong polynomial* for β.

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$$(b_k b_{k-1} \cdots b_1 b_0 . b_{-1} \cdots b_{-h})_\beta = 0$$

Suppose that β is SRZ, i.e. $B > 2M$.
Working alphabet $\mathcal{A} = \{-a, \ldots, 0, \ldots, a\}$
with

$$a = \left\lceil \frac{B-1}{2} \right\rceil + \left\lceil \frac{B-1}{2(B-2M)} \right\rceil M.$$

Let

$$a' = \left\lceil \frac{B-1}{2} \right\rceil \quad \text{and} \quad c = \left\lceil \frac{B-1}{2(B-2M)} \right\rceil.$$

Then $a = a' + cM$.
$\mathcal{A}' = \{-a', \ldots, 0, \ldots, a'\} \subset \mathcal{A}$ is the inner alphabet.

Parallel addition for base β SRZ on

$$\mathcal{A} = \{-a, \ldots, 0, \ldots, a\},\ a = \underbrace{\left\lceil \frac{B-1}{2} \right\rceil}_{a'} + \underbrace{\left\lceil \frac{B-1}{2(B-2M)} \right\rceil}_{c} M$$

Algorithm (S)

Input: $x_n \cdots x_m$ and $y_n \cdots y_m$ in $\mathcal{A}^*$, with $m \leqslant n$,
$x = \sum_{i=m}^{n} x_i \beta^i$ and $y = \sum_{i=m}^{n} y_i \beta^i$.
Output: $z_{n+k} \cdots z_{m-h}$ in $\mathcal{A}^*$ such that $z = x + y = \sum_{i=m-h}^{n+k} z_i \beta^i$.

`for each` i `in parallel do`

0. $z_i := x_i + y_i$
1. find $q_i \in \{-c, \ldots, 0, \ldots, c\}$ such that $z_i - q_i B \in \mathcal{A}'$
2. $z_i := z_i - \sum_{j=-h}^{k} q_{i-j} b_j$

Parallel addition for base β SRZ on

$$\mathcal{A} = \{-a, \dots, 0, \dots, a\},\ a = \underbrace{\left\lceil \frac{B-1}{2} \right\rceil}_{a'} + \underbrace{\left\lceil \frac{B-1}{2(B-2M)} \right\rceil}_{c} M$$

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Output: $z_{n+k} \cdots z_{m-h}$ in $\mathcal{A}^*$ such that $z = x + y = \sum_{i=m-h}^{n+k} z_i \beta^i$.

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0. $z_i := x_i + y_i$
1. find $q_i \in \{-c, \dots, 0, \dots, c\}$ such that $z_i - q_i B \in \mathcal{A}'$
2. $z_i := z_i - \sum_{j=-h}^{k} q_{i-j} b_j$

Algorithm (S) is neighbour free.

Integer base

$\beta = b$ integer $\geqslant 3$ is SRZ for the polynomial $-X + b$, and Algorithm (S) works with $c = 1$, $a' = \lceil \frac{b-1}{2} \rceil$, and $a = \lceil \frac{b+1}{2} \rceil$.

`for each` i `in parallel do`

0. $z_i := x_i + y_i$
1. find $q_i \in \{-1, 0, 1\}$ such that $z_i - q_i b \in \mathcal{A}'$
2. $z_i := z_i - q_i b + q_{i-1}$

Algorithm (S) is the algorithm of Avizienis with $a = \lceil \frac{b+1}{2} \rceil$.

For $\beta = 2$, $-X + 2$ is not a strong polynomial. But β satisfies the strong polynomial

$$-X^2 + 4.$$

So Algorithm (S) works for base 2 on $\{-3, \ldots, 0, \ldots, 3\}$.

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So Algorithm (S) works for base 2 on $\{-3, \ldots, 0, \ldots, 3\}$.

Remind that the Chow and Robertson algorithm works with smaller alphabet $\{-1, 0, 1\}$, but need to examine the right neighbour of current position.

The Golden Mean

$\beta = \frac{1+\sqrt{5}}{2}$, the Golden Mean.
Every real number $\geqslant 0$ has an expansion on alphabet $\{0, 1\}$.

β is one root of $X^2 - X - 1$, the second root is $\beta' = \frac{1-\sqrt{5}}{2} = -\frac{1}{\beta}$.
Since $\beta^4 + (\beta')^4 = 7$, β is a root of the strong polynomial

$$S(X) = -X^4 + 7 - \frac{1}{X^4}$$

with $B = 7$ and $M = 2$. Thus $c = 1$, $a' = 3$, and $a = 5$. The working alphabet of Algorithm (S) is $\mathcal{A} = \{-5, \ldots, 0, \ldots, 5\}$.

$\bar{1}0007000\bar{1}$ is a strong β-representation of 0.

$a' = 3$, $a = 5$

x	$\mapsto$						2	5	$\bar{2}$	5	$\bar{5}$	0	0	3				
y	$\mapsto$					5	1	2	$\bar{2}$	5	$\bar{4}$	0	0	5				
z	$\mapsto$					5	3	7	$\bar{4}$	10	$\bar{9}$	0	0	8				
0	$\mapsto$	1	0	0	0	$\bar{7}$	0	0	0	1								
0	$\mapsto$			1	0	0	0	$\bar{7}$	0	0	0	1						
0	$\mapsto$				$\bar{1}$	0	0	0	7	0	0	0	$\bar{1}$					
0	$\mapsto$					1	0	0	0	$\bar{7}$	0	0	0	1				
0	$\mapsto$						$\bar{1}$	0	0	0	7	0	0	0	$\bar{1}$			
0	$\mapsto$									1	0	0	0	$\bar{7}$	0	0	0	1
z	$\mapsto$	1	0	1	$\bar{1}$	$\bar{1}$	2	0	3	5	$\bar{2}$	1	$\bar{1}$	2	$\bar{1}$	0	0	1

Gaussian integer

Example

Base $\beta = -1 + i$.
Every complex number has an expansion on alphabet $\{0, 1\}$.

β satisfies the strong polynomial $X^4 + 4$ with $B = 4$ and $M = 1$.
The working alphabet of Algorithm (S) is
$\mathcal{A} = \{-3, \ldots, 0, \ldots, 3\}$.

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β satisfies the strong polynomial $X^4 + 4$ with $B = 4$ and $M = 1$.

The working alphabet of Algorithm (S) is

$\mathcal{A} = \{-3, \ldots, 0, \ldots, 3\}$.

On $\{-1, 0, 1\}$, there is a parallel addition algorithm provided by Nielsen and Muller (1996).

Rational base

Example

$\beta = \frac{7}{2}$ is an algebraic number which is not an algebraic integer. β is SRZ for the polynomial $S(X) = -2X + 7$ with $B = 7$ and $M = 2$. Thus $a' = 3$, $c = 1$, and $a = 5$.

Any integer has a finite representation on the redundant alphabet $\{-5, \ldots, 0, \ldots, 5\}$ (Frougny and Klouda 2010), and, by Algorithm (S), addition is realizable in parallel.

Locality

Corollary

If β is SRZ with strong polynomial $S(X) = b_k X^k + b_{k-1} X^{k-1} + \cdots + b_1 X + b_0 + b_{-1} X^{-1} + \cdots + b_{-h} X^{-h}$ then addition realized by Algorithm (S) is a $(h + k + 1)$-local function from $\{-2a, \ldots, 0, \ldots, 2a\}^{\mathbb{Z}}$ to $\mathcal{A}^{\mathbb{Z}}$.

What numbers are SRZ?

Proposition

Let β with $|\beta| > 1$ be an algebraic number such that all its algebraic conjugates have modulus $\neq 1$. Then β is SRZ.

The proof gives a constructive method to obtain a strong polynomial from the minimal polynomial of β.

What numbers are SRZ?

Proposition

Let β with $|\beta| > 1$ be an algebraic number such that all its algebraic conjugates have modulus $\neq 1$. Then β is SRZ.

The proof gives a constructive method to obtain a strong polynomial from the minimal polynomial of β.

Theorem

Let β with $|\beta| > 1$ be an algebraic number of degree d.

- *If $d = 2$, then β is SRZ.*
- *If d is even $\geqslant 4$ and the minimal polynomial of β is not reciprocal, then β is SRZ.*
- *If d is odd, then β is SRZ.*

Reduction of the alphabet

Definition

β satisfies the *weak representation of zero property* (β *is WRZ*) if there exist integers $b_k, b_{k-1}, \ldots, b_1, b_0, b_{-1}, \ldots, b_{-h}$ with h and k non-negative integers, such that β is a root of the polynomial

$$W(X) = b_k X^k + b_{k-1} X^{k-1} + \ldots + b_1 X + b_0 + b_{-1} X^{-1} + \ldots + b_{-h} X^{-h}$$

and

$$B = b_0 > \sum_{i \neq 0} |b_i| = M\,.$$

The polynomial W is said to be a *weak polynomial* for β.

β is WRZ, i.e. $B > M$. Working alphabet

$$\mathcal{A} = \{-a, \ldots, 0, \ldots, a\}, \quad \text{where} \quad a = \left\lceil \frac{B-1}{2} \right\rceil + M\,.$$

Inner alphabet is $\mathcal{A}' = \{-a', \ldots, 0, \ldots, a'\}$ with $a' = \lceil \frac{B-1}{2} \rceil$.

Algorithm (W) works with $\left\lceil \frac{a}{B-M} \right\rceil$ iterations.

Parallel addition for base β WRZ on $\mathcal{A} = \{-a, \ldots, 0, \ldots, a\}$, $a = \underbrace{\lceil \frac{B-1}{2} \rceil}_{a'} + M$

Algorithm (W)

Input: $x_n \cdots x_m$ and $y_n \cdots y_m$ in $\mathcal{A}^*$, with $m \leqslant n$,
$x = \sum_{i=m}^{n} x_i \beta^i$ and $y = \sum_{i=m}^{n} y_i \beta^i$.
Output: $z_{n+k} \cdots z_{m-h}$ in $\mathcal{A}^*$ such that

$$z = x + y = \sum_{i=m-h}^{n+k} z_i \beta^i.$$

for each i in parallel do

0. $z_i := x_i + y_i$
1. for $\ell := 1$ to $\left\lceil \frac{a}{B-M} \right\rceil$ do
   if $z_i \in \mathcal{A}'$ then $q_i := 0$ else $q_i := \operatorname{sgn} z_i$
   $z_i := z_i - \sum_{j=-h}^{k} q_{i-j} b_j$

Example

$\beta = \frac{1+\sqrt{5}}{2}$, the Golden Mean.

Since $-\beta^2 + 3 - \frac{1}{\beta^2} = 0$, β is a root of the weak polynomial

$$W(X) = -X^2 + 3 - \frac{1}{X^2}$$

with $B = 3$ and $M = 2$. Thus $a' = 1$, and $a = 3$.

Algorithm (W) works on $\mathcal{A} = \{-3, \ldots, 0, \ldots, 3\}$, with 3 iterations.

$\bar{1}030\bar{1}$ is a weak β-representation of 0.

Step 0.	x	$\mapsto$					3	$\bar{1}$	3	0	3				
	y	$\mapsto$					2	0	3	$\bar{2}$	3				
	z	$\mapsto$					5	$\bar{1}$	6	$\bar{2}$	6				
Step 1.	0	$\mapsto$			1	0	$\bar{3}$	0	1						
	0	$\mapsto$					1	0	$\bar{3}$	0	1				
	0	$\mapsto$						$\bar{1}$	0	3	0	$\bar{1}$			
	0	$\mapsto$							1	0	$\bar{3}$	0	1		
	z	$\mapsto$			1	0	3	$\bar{2}$	5	1	4	$\bar{1}$	1		
Step 2.	0	$\mapsto$			1	0	$\bar{3}$	0	1						
	0	$\mapsto$				$\bar{1}$	0	3	0	$\bar{1}$					
	0	$\mapsto$					1	0	$\bar{3}$	0	1				
	0	$\mapsto$							1	0	$\bar{3}$	0	1		
	z	$\mapsto$			2	$\bar{1}$	1	1	4	0	2	$\bar{1}$	2		
Step 3.	0	$\mapsto$	1	0	$\bar{3}$	0	1								
	0	$\mapsto$					1	0	$\bar{3}$	0	1				
	0	$\mapsto$							1	0	$\bar{3}$	0	1		
	0	$\mapsto$									1	0	$\bar{3}$	0	1
	z	$\mapsto$	1	0	$\bar{1}$	$\bar{1}$	3	1	2	0	1	$\bar{1}$	0	0	1

Corollary

If β is WRZ with weak polynomial $W(X) = b_k X^k + b_{k-1} X^{k-1} + \cdots + b_1 X + b_0 + b_{-1} X^{-1} + \cdots + b_{-h} X^{-h}$ then addition realized by Algorithm (W) is a $(h \left\lceil \frac{a}{B-M} \right\rceil + k \left\lceil \frac{a}{B-M} \right\rceil + 1)$-local function from $\{-2a, \ldots, 0, \ldots, 2a\}^{\mathbb{Z}}$ to $\mathcal{A}^{\mathbb{Z}}$.

Corollary

If β is WRZ with weak polynomial $W(X) = b_k X^k + b_{k-1} X^{k-1} + \cdots + b_1 X + b_0 + b_{-1} X^{-1} + \cdots + b_{-h} X^{-h}$ then addition realized by Algorithm (W) is a $\left(h \left\lceil \frac{a}{B-M} \right\rceil + k \left\lceil \frac{a}{B-M} \right\rceil + 1\right)$-local function from $\{-2a, \ldots, 0, \ldots, 2a\}^{\mathbb{Z}}$ to $\mathcal{A}^{\mathbb{Z}}$.

Algorithm (W) is neighbour free.

Corollary

If β is WRZ with weak polynomial $W(X) = b_k X^k + b_{k-1} X^{k-1} + \cdots + b_1 X + b_0 + b_{-1} X^{-1} + \cdots + b_{-h} X^{-h}$ then addition realized by Algorithm (W) is a $(h\left\lceil \frac{a}{B-M} \right\rceil + k\left\lceil \frac{a}{B-M} \right\rceil + 1)$-local function from $\{-2a, \ldots, 0, \ldots, 2a\}^{\mathbb{Z}}$ to $\mathcal{A}^{\mathbb{Z}}$.

Algorithm (W) is neighbour free.

Remark

Algorithm (S) and Algorithm (W) coincide if, and only if, $B \geqslant 4M - 1$.

Example

- If b integer $\geqslant 3$, $-X + b$ is a strong polynomial for b, with $B = b$ and $M = 1$, thus $B \geqslant 4M - 1$. Algorithm (S) and Algorithm (W) coincide with $\mathcal{A} = \{-a, \ldots, a\}$, $a = \left\lceil \frac{b+1}{2} \right\rceil$.
- For $b = 2$, $-X + 2$ is a weak polynomial. Algorithm (W) works with $\mathcal{A} = \{-2, -1, 0, 1, 2\}$ with 2 iterations.

The Golden Mean

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We give Algorithm (G) for parallel addition in base the Golden Mean on $\{-1, 0, 1\}$. This algorithm is neighbour sensitive.

It is not possible to perform parallel addition in base the Golden Mean on $\{0, 1\}$.

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We give Algorithm (G) for parallel addition in base the Golden Mean on $\{-1, 0, 1\}$. This algorithm is neighbour sensitive.

It is not possible to perform parallel addition in base the Golden Mean on $\{0, 1\}$.

We use the weak representation of zero $-\beta^2 + 3 - \frac{1}{\beta^2} = 0$.

Algorithm A: Base $\beta = \frac{1+\sqrt{5}}{2}$, reduction from $\{-2, -1, 0, 1, 2\}$ to $\{-1, 0, 1, 2\}$.

Input: a finite sequence of digits (z_i) of $\{-2, -1, 0, 1, 2\}$, with $z = \sum z_i \beta^i$.
Output: a finite sequence of digits (z_i) of $\{-1, 0, 1, 2\}$, with $z = \sum z_i \beta^i$.

`for each` i `in parallel do`

1. `case` $\left\{ \begin{array}{l} z_i = -2 \\ z_i = -1 \\ z_i = 0 \text{ and } z_{i+2} < 0 \text{ and } z_{i-2} < 0 \end{array} \right\}$ `then` $q_i := -1$

   `else` $q_i := 0$

2. $z_i := z_i - 3q_i + q_{i+2} + q_{i-2}$

Algorithm B: Base $\beta = \frac{1+\sqrt{5}}{2}$, reduction from $\{-1, 0, 1, 2\}$ to $\{-1, 0, 1\}$.

Input: a finite sequence of digits (z_i) of $\{-1, 0, 1, 2\}$, with $z = \sum z_i \beta^i$.
Output: a finite sequence of digits (z_i) of $\{-1, 0, 1\}$, with $z = \sum z_i \beta^i$.

for each i in parallel do
1. case
$$\left\{\begin{array}{l} z_i = 2 \\ z_i = 1 \text{ and } (z_{i+2} \geqslant 1 \text{ or } z_{i-2} \geqslant 1) \\ z_i = 0 \text{ and } z_{i+2} = z_{i-2} = 2 \\ z_i = 0 \text{ and } z_{i+2} = z_{i-2} = 1 \text{ and } z_{i+4} \geqslant 1 \text{ and } z_{i-4} \geqslant 1 \\ z_i = 0 \text{ and } z_{i+2} = 2 \text{ and } z_{i-2} = 1 \text{ and } z_{i-4} \geqslant 1 \\ z_i = 0 \text{ and } z_{i-2} = 2 \text{ and } z_{i+2} = 1 \text{ and } z_{i+4} \geqslant 1 \end{array}\right\}$$
then $q_i := 1$

else $q_i := 0$

2. $z_i := z_i - 3q_i + q_{i+2} + q_{i-2}$

Algorithm (G): Base $\beta = \frac{1+\sqrt{5}}{2}$, parallel addition on $\mathcal{A} = \{-1, 0, 1\}$.

Input: two finite sequences of digits (x_i) and (y_i) of $\{-1, 0, 1\}$, with $x = \sum x_i \beta^i$ and $y = \sum y_i \beta^i$.
Output: a finite sequence of digits (z_i) of $\{-1, 0, 1\}$ such that

$$z = x + y = \sum z_i \beta^i.$$

`for each` i `in parallel do`

0. $v_i := x_i + y_i$
1. use Algorithm A with input (v_i) and output (w_i)
2. use Algorithm B with input (w_i) and output (z_i)

Corollary

Addition in base the Golden Mean realized by Algorithm (G) is a 21-local function from $\{-2, -1, 0, 1, 2\}^{\mathbb{Z}}$ *to* $\{-1, 0, 1\}^{\mathbb{Z}}$.

Some questions

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  If $\beta = b$ odd integer $\geqslant 3$, $a = \lceil \frac{b+1}{2} \rceil$ (Avizienis).
  If $\beta = b$ even integer $\geqslant 2$, $a = \frac{b}{2}$ (Chow and Robertson).
  If β is the Golden Mean, $a = 1$.

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- What is the minimal symmetric alphabet $\mathcal{A} = \{-a, \ldots, 0, \ldots, a\}$ for parallel addition?
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  If $\beta = b$ even integer $\geqslant 2$, $a = \frac{b}{2}$ (Chow and Robertson).
  If β is the Golden Mean, $a = 1$.
- What are the minimal alphabets for parallel addition?

Some questions

$\mathcal{A}$ alphabet of contiguous integer digits.

- What is the minimal symmetric alphabet $\mathcal{A} = \{-a, \ldots, 0, \ldots, a\}$ for parallel addition?
  If $\beta = b$ odd integer $\geqslant 3$, $a = \lceil \frac{b+1}{2} \rceil$ (Avizienis).
  If $\beta = b$ even integer $\geqslant 2$, $a = \frac{b}{2}$ (Chow and Robertson).
  If β is the Golden Mean, $a = 1$.
- What are the minimal alphabets for parallel addition?
  If $\beta = b \geqslant 2$, an alphabet of cardinality $b + 1$:
  $\{-1, 0, \ldots, b-1\}$, or $\{0, \ldots, b-1, b\}$, ...